WORCESTER COUNTY MATHEMATICS LEAGUE

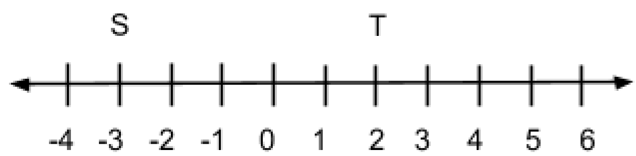
Freshman Meet 3 - February 11, 2015
Round 1: Graphing on a Number Line

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Graph the inequality -4 < 2x - 3 < 8 on a number line. Specify non-integer endpoints.

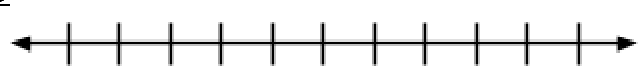
2. Given



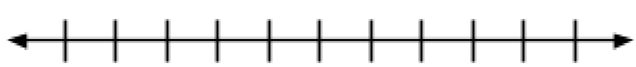
Find the coordinate of the point between S and T that is twice the distance from S as it is from T.

3. Graph the following solution set: $4 - x \le |5x - 2| \le x + 2$

ANSWERS

(1 pt.) 1. 

(2 pts.) 2. ________________________________

(3 pts.) 3. 

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 11, 2015
Round 2: Operations on Polynomials

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Simplify: $(2x^2 - 3xy)(3x - y)$

2. Assume $4m \neq 3n$. Simplify: $(24m^2 - 14mn - 3n^2) \div (4m - 3n)$.

3. Solve $x^4 + 10x^3 + 25x^2 - 36 = 0$.

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________

(3 pts.) 3. ______________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 11, 2015
Round 3: Techniques of Counting and Probability

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A fair coin is flipped 4 times. What is the probability that a string of at least 3 consecutive heads or a string of at least 3 consecutive tails appears?

2. Seven red flags, two yellow flags, and one green flag are to be displayed in a single line at a ceremony. How many distinct ways can these flags be arranged?

3. Consider the set of integers {1, 2, …, 2015}. If we draw one of these integers randomly, what is the probability that it will be a multiple of 2, 3, or 5? **(do not attempt to simplify the answer)**

ANSWERS

(1 pt.) 1. ________________________________

(2 pts.) 2. ________________________________

(3 pts.) 3. ________________________________

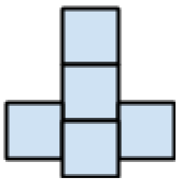
WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 11, 2015
Round 4: Perimeter, Area, and Volume

All answers must be in simplest exact form in the answer section

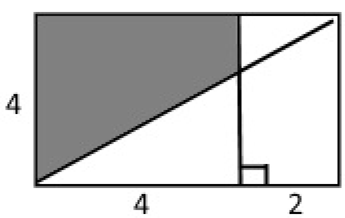
NO CALCULATOR ALLOWED

1. Suppose the figure below is constructed using five identical square tiles and that the area of the figure is 245 sq. units. What is the perimeter of this figure?



2. Suppose the ratio of the base of triangle A to the base of triangle B is 2:3. Further, suppose the ratio of the area of A to the area of B is 3:5. Find the ratio of the altitude of A to the altitude of B.

3. Find the area of the shaded part of the rectangle below:



ANSWERS

(1 pt.) 1. ______________________ units

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________ sq. units

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 11, 2015
Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. Suppose that $\frac{25-3|x|}{-(2x+5)}>0$. Graph on a number line the possible values of x.

2. Factor completely: $(4-4x-x^2)(4-4x+3x^2)+4x^4$.

3. A box contains 5 rods measuring 20, 36, 45, 60, and 90 units in length. Suppose three rods are randomly chosen from the box. What is the probability that they can be arranged to form a triangle?

4. Consider a rectangular solid that is made up of $\frac{1}{2}$" cubes stacked together. The solid has a length of 5", and its height is 1.5 times its width. If the total surface area of all the individual cubes used to form the solid is 1440 sq. inches, what is the height of the solid, in inches?

5. Set A has 56 more subsets than set B. How many more elements are in A than in B?

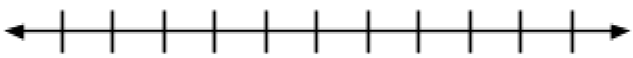
6. Find all positive integral solutions for $3x+5y=79$. Express the solutions as (x, y) ordered pairs.

7. Find 3 consecutive even numbers such that if they are divided by 2, 4, and 6 respectively, the sum of their quotients will equal the next higher consecutive even number.

8. The degree measure of one of two complementary angles is 30 less than twice that of the other. What are the degree measures of the angles?

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 2 - December 10, 2014
Team Round Answer Sheet

1. 

2. ____________________________

3. ____________________________

4. ____________________________ inches

5. ____________________________

6. ____________________________ %

7. ____________________________

8. ____________________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Freshman Meet 3 - February 11, 2015 SOLUTIONS

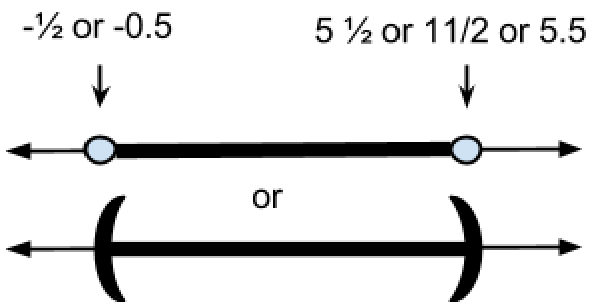
Round 1: Graphing on a Number Line

1. Graph the inequality -4 < 2x - 3 < 8 on a number line. Specify non-integer endpoints.

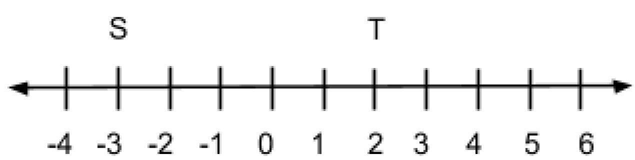
SOLUTION: Begin by isolating x in this inequality:

$-4 < 2x - 3 < 8$
$-4 + 3 < 2x < 8 + 3$
$-1 < 2x < 11$
$-\frac{1}{2} < x < 5\frac{1}{2}$

Now we graph the solution set:



2. Given



Find the coordinate of the point between S and T that is twice the distance from S as it is from T.

SOLUTION: We have that S is at point -3 and T is at point 2, which means the distance between them is 5 units. We need the desired point to be precisely two thirds of this distance away from S, or $\frac{2}{3} \times 5 = \frac{10}{3}$ units away from S. Therefore, the coordinate of the desired point should be equal to $-3 + \frac{10}{3} = \frac{1}{3}$.

3. Graph the following solution set: $4-x \le |5x-2| \le x+2$

SOLUTION: This solution set can be rewritten into an AND statement: $4-x \le |5x-2|$ AND $|5x-2| \le x+2$. We will approach each side of this AND statement individually.

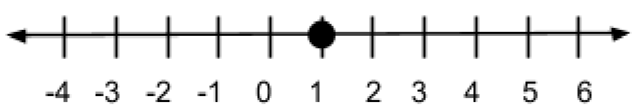
Let's start by examining the inequality $4-x \le |5x-2|$. Since the quantity contained in the absolute value is greater than the value on the left hand side, we know that we can rewrite this inequality into an OR statement: $4-x \le 5x-2$ OR $x-4 \ge 5x-2$. Now let's solve each side of this OR statement.

$4-x \le 5x-2$	OR	$x-4 \ge 5x-2$
$6 \le 6x$	OR	$-2 \ge 4x$
$1 \le x$	OR	$-\frac{1}{2} \ge x$

Now we are ready to tackle the second part of the original AND statement, which was $|5x-2| \le x+2$. Since the absolute value here is less than the quantity on the right hand side, we know that we can rewrite this into an AND statement: $5x-2 \le x+2$ AND $5x-2 \ge -x-2$. Now let's solve each side of this AND statement.

$5x-2 \le x+2$	AND	$5x-2 \ge -x-2$
$4x \le 4$	AND	$6x \ge 0$
$x \le 1$	AND	$x \ge 0$

Putting everything together now, we have that ($1 \le x$ OR $-\frac{1}{2} \ge x$) AND ($x \le 1$ AND $x \ge 0$). Since x must be both weakly less than 1 and weakly greater than 1, we have that the solution set is x = 1, or



Round 2: Operations on Polynomials

1. Simplify: $(2x^2-3xy)(3x-y)$

SOLUTION: We have that

$$(2x^2-3xy)(3x-y) =$$
$$6x^3-2x^2y-9x^2y+3xy^2 =$$
$$6x^3-11x^2y+3xy^2$$

2. Assume $4m \ne 3n$. Simplify: $(24m^2-14mn-3n^2) \div (4m-3n)$

Solution: We will begin by factoring the dividend:

$$24m^2 - 14mn - 3n^2$$
$$(4m - 3n)(6m + n)$$

The expression then becomes $(4m - 3n)(6m + n) \div (4m - 3n) = (6m + n)$.

3. Solve $x^4 + 10x^3 + 25x^2 - 36 = 0.$

Solution:

$$x^4 + 10x^3 + 25x^2 - 36 = 0$$
$$x^2(x^2 + 10x + 25) - 36 = 0$$
$$x^2(x+5)^2 - 36 = 0$$
$$(x(x+5))^2 - 6^2 = 0$$
$$(x(x+5) + 6)(x(x+5) - 6) = 0$$
$$(x^2 + 5x + 6)(x^2 + 5x - 6) = 0$$
$$(x + 3)(x + 2)(x + 6)(x - 1) = 0$$

Therefore, the solution set is x = -6, -3, -2, 1

Round 3: Techniques of Counting and Probability

1. A fair coin is flipped 4 times. What is the probability that a string of at least 3 consecutive heads or a string of at least 3 consecutive tails appears?

Solution: We have that there are six ways the event described in the problem can occur:

HHHH	TTTT
HHHT	TTTH
TTTH	HTTT

The total number of possible outcomes is given by $2^4 = 16.$ Therefore the probability of the event described in the problem is $\frac{6}{16} = \frac{3}{8}$.

2. Seven red flags, two yellow flags, and one green flag are to be displayed in a single line at a ceremony. How many distinct ways can these flags be arranged?

Solution: We have that there are $\binom{10}{7}$ ways of arranging the seven red flags. After arranging the red flags, there are only 3 spots left, which means there are $\binom{3}{2}$ ways of arranging the yellow flags. Finally there is only one open spot remaining so there is $\binom{1}{1}$ way of arranging the one green flag. Multiplying these together gives

$$\binom{10}{7}\binom{3}{2}\binom{1}{1} =$$
$$\left(\frac{10!}{7!3!}\right)\left(\frac{3!}{2!1!}\right)\left(\frac{1!}{1!0!}\right) =$$
$$\frac{10\times9\times8}{2} =$$
$$90\times4 = 360.$$

Note: we can perform this same flavor of solution using a different order of placing flags, reaching an equivalent answer. For instance, there are $\binom{10}{1}$ ways of arranging the one green flag. In the remaining nine spots, there are $\binom{9}{2}$ ways of arranging the yellow flags and then $\binom{7}{7}$ ways of arranging the last seven red flags. Multiplying these terms together reaches the same answer of 360 arrangements.

3. Consider the set of integers {1, 2, …, 2015}. If we draw one of these integers randomly, what is the probability that it will be a multiple of 2, 3, or 5? **(do not attempt to simplify the answer)**

Solution: Let's begin by examining the integers 1, 2, …, 30. In this set, we have:

6 multiples of 5:	5, 10, 15, 20, 25, 30
10 multiples of 3:	3, 6, 9, 12, 15, 18, 21, 24, 27, 30
15 multiples of 2:	2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30

3 numbers that are multiples of both 5 and 2:	10, 20, 30
2 numbers that are multiples of both 5 and 3:	15, 30
5 numbers which are multiples of both 2 and 3:	6, 12, 18, 24, 30

1 number which is a multiple of 2, 3, and 5:	30

Using the Inclusion-exclusion principle, we have the number of integers in the set {1, 2, …, 30} which are multiples of either 2, 3, or 5 is given by:

$$(6 + 10 + 15) - (3 + 2 + 5) + (1) = 22.$$

Now note that this pattern of multiples repeats every 30 digits after the set {1, 2, …, 30}. We have that 2010/30 = 67. Therefore, we need only multiply 22 by 67 to see how many multiples of 2, 3, or 5 are contained in the set {1, 2, …, 2010}. 22 x 67 = 1474.

The only remaining digits we need to account for are 2011, …, 2015. In this set there are two multiples of 2, one multiple of 3, and one multiple of 5, none of which overlap. Therefore, there are 1478 integers in the set which are multiples of 2, 3, or 5. Hence, the probability of randomly drawing one such number from the grand set is given by 1478/2015.

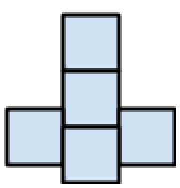
Alternate Solution: Solving directly, we have that there are

1007 terms divisible by 2
671 terms divisible by 3
403 terms divisible by 5
335 terms divisible by 6
201 terms divisible by 10
134 terms divisible by 15
67 terms divisible by 30

so (1007 + 671 + 403) - (335 + 201 + 134) + 67 = 1478. Therefore, we have 1478/2015.

Round 4: Perimeter, Area, and Volume

1. Suppose the figure below is constructed using five identical square tiles and that the area of the figure is 245 sq. units. What is the perimeter of this figure?



Solution: Dividing 245 by 5 gives us that each tile has an area of 49 sq. units. This means each tile has 7 unit long sides. We can then count that there are precisely 12 7-unit length sides on the outside of the figure (in the perimeter). Hence, the perimeter is given by 12 x 7 = 84 units.

2. Suppose the ratio of the base of triangle A to the base of triangle B is 2:3. Further, suppose the ratio of the area of A to the area of B is 3:5. Find the ratio of the altitude of A to the altitude of B.

Solution: Denote the bases of the triangles by $2k$ and $3k$ and the altitudes by h_1 and h_2, respectively. Since the ratio of the areas is given by 3:5, we have that

$$\frac{\frac{1}{2}\times 2k\times h_A}{\frac{1}{2}\times 3k\times h_B}=\frac{3}{5}$$
$$\frac{2h_A}{3h_B}=\frac{3}{5}$$
$$\frac{h_A}{h_B}=\frac{9}{10}$$

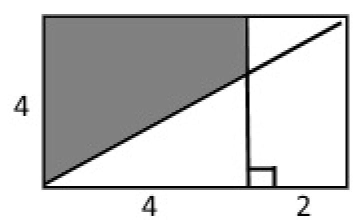
Therefore, the ratio of the altitude of A to the altitude of B is 9:10.

Alternate solution: Assume k = 1. This gives that the first triangle has base 2 and altitude x and the second triangle has base 3 and altitude y.

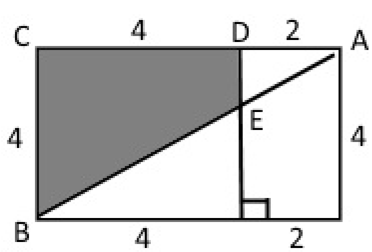
The area formula gives:

$$\frac{\frac{1}{2}\times 2\times x}{\frac{1}{2}\times 3\times y}=\frac{3}{5}$$
$$\frac{2x}{3y}=\frac{3}{5}$$
$$\frac{x}{y}=\frac{9}{10}$$

3. Find the area of the shaded part of the rectangle below:



Solution:



Note that the shaded region in this figure is a trapezoid with $b_2 = 4, h = 4.$ We need to figure out the length of $b_1,$ which is represented by segment DE.

Notice that △ABC and △ADE are similar, so we have that
$$\frac{2}{DE}=\frac{6}{4}$$
$$DE=\frac{8}{6}=\frac{4}{3}$$

Therefore, the area of the shaded region is given by $\frac{1}{2}\left(\frac{4}{3}+4\right)(4)=\frac{1}{2}\left(\frac{16}{3}\right)(4)=\frac{32}{3}$ sq. units.

Alternate Solution: using the same method as above we find that DE = $\frac{4}{3}$. Therefore, to find the area of the shaded region, we need only subtract the area of △ADE from the area of △ABC. This translates into $\frac{1}{2}(6\times 4)-\frac{1}{2}(2\times\frac{4}{3})=12-\frac{4}{3}=\frac{32}{3}$ sq. units.

Team Round

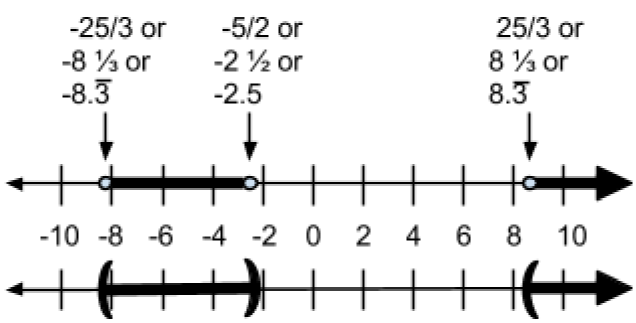
1. Suppose that $\frac{25-3|x|}{-2x-5}>0$. Graph on a number line the possible values of x.

Solution: We can solve this problem by determining the set of values where the numerator and denominator are both positive or both negative.

Numerator Positive: $|x|<\frac{25}{3}$. This means we need all the points less than $\frac{25}{3}$ units away from 0 on the number line. This set is expressed as $\{-\frac{25}{3}<x<\frac{25}{3}\}$
Denominator Positive: $x<-\frac{5}{2}$
Both Positive: $\{-\frac{25}{3}<x<\frac{25}{3}\}\cap\{x<-\frac{5}{2}\}=\{-\frac{25}{3}<x<-\frac{5}{2}\}$

Numerator Negative: $|x|>\frac{25}{3}$. This means we need all the points greater than $\frac{25}{3}$ units away from 0 on the number line. This set is expressed as $\{x<-\frac{25}{3}\}\cup\{x>\frac{25}{3}\}$
Denominator Negative: $x>-\frac{5}{2}$
Both Negative: $(\{x<-\frac{25}{3}\}\cup\{x>\frac{25}{3}\})\cap\{x>-\frac{5}{2}\}=\{x>\frac{25}{3}\}$

Therefore, the solution is given by the union of these two sets: $\{-\frac{25}{3}<x<-\frac{5}{2}\}\cup\{x>\frac{25}{3}\}$.
Graphically, this solution is given by:



2. Factor completely: $(4-4x-x^2)(4-4x+3x^2)+4x^4$

Solution:

$$(4-4x-x^2)(4-4x+3x^2)+4x^4$$
$$(4-4x+x^2-2x^2)(4-4x+x^2+2x^2)+4x^4$$
$$((x-2)^2-2x^2)((x-2)^2+2x^2)+4x^4$$
$$(x-2)^4-(2x^2)^2+4x^4$$
$$(x-2)^4-4x^4+4x^4=(x-2)^4$$

3. A box contains 5 rods measuring 20, 36, 45, 60, and 90 units in length. Suppose three rods are randomly chosen from the box. What is the probability that they can be arranged to form a triangle?

Solution: By the Triangle Inequality Theorem we have that there are five possible rod combinations which can form a triangle:

{20, 36, 45}, {20, 45, 60}, {36, 45, 60}, {36, 60, 90} and {45, 60, 90}

The number of ways we can select 3 rods from 5 is given by $\binom{5}{3}=\frac{5!}{3!2!}=\frac{5\times4}{2}=10$
Therefore the probability of forming a triangle is $\frac{5}{10}=\frac{1}{2}$ or 0.5.

4. Consider a rectangular solid that is made up of $\frac{1}{2}$" cubes stacked together. The solid has a length of 5”, and its height is 1.5 times its width. If the total surface area of all the individual cubes used to form the solid is 1440 sq. inches, what is the height of the solid, in inches?

Solution: We are given that $h = 1.5w$. Therefore, the number of $\frac{1}{2}$" cubes needed in order to build this solid is $(5\times 2)\times(w\times 2)\times(1.5w\times 2)=60w^2$. Now note that each cube has a surface area of $6\times\frac{1}{2}\times\frac{1}{2}=1.5$ sq. inches. Therefore, the number of cubes in the solid is given by 1440/1.5 = 960. This now allows us to form the equation

$$60w^2 = 960$$
$$w^2 = 16$$
$$w = 4"$$

Now that we know the width of the solid, we have that the height is 1.5w = 6”.

Alternate solution: The surface area of one cube is $6\times\frac{1}{2}\times\frac{1}{2}=\frac{6}{4}=\frac{3}{2}$ sq. inches. So we know that there are a total of $1440\div\frac{3}{2}$ = 960 cubes. Now the volume of a rectangular prism is given by V = bhw. Since there are 10 cubes in one dimension (base), we know that the # of cubes in the width width times the # of cubes in the height = 96. From the given information we have that $h = 1.5w$, so then $w\times(1.5w)=96 \Rightarrow w^2=64 \Rightarrow w=8$ (this is the number of blocks in the width). Therefore the height is 12 blocks and 12 blocks is 6 inches.

5. Set A has 56 more subsets than set B. How many more elements are in set A than set B?

Solution: We know that for a set with n elements it has 2^n subsets. Let's create a table showing for every size set how many subsets that set contains:

# of Elements	# of Subsets
0	1
1	2
2	4
3	8
4	16
5	32
6	64

We know that Set A has 56 more subsets than set B. Looking in the table we see that 64 - 8 = 56. Therefore, set A must have 6 elements and set B must have 3 elements, meaning that set A has 3 more elements than set B.

6. Find all positive integral solutions for $3x + 5y = 79$. Express the solutions as (x, y) ordered pairs.

Solution: Begin by noting that we need the units digit to end in 9. Using the sum of multiples of 3 and 5, there are only two ways to make this happen: either the multiple of 3 ends in 9 and the multiple of 5 ends in 0, or the multiple of 3 ends in 4 and the multiple of 5 ends in 5.

<u>Multiples of 3 ending in 9 (which are smaller than 79)</u>: 9, 39, 69
<u>Multiples of 5 ending in 0 (which are smaller than 79)</u>: 10, 20, 30, 40, 50, 60, 70

From these lists we see that 3 combinations work: 9+70, 39+40, and 69+10. The (x,y) solutions corresponding to these combinations are (3, 14), (13, 8), and (23, 2).

<u>Multiples of 3 ending in 4 (which are smaller than 79)</u>: 24, 54
<u>Multiples of 5 ending in 5 (which are smaller than 79)</u>: 5, 15, 25, 35, 45, 55, 65, 75

From these lists we see that 2 combinations work: 24+55 and 54+25. The (x,y) solutions corresponding to these combinations are (8, 11) and (18, 5).

Therefore, the solution set is (3, 14), (8, 11), (13, 8), (18, 5), and (23, 2).

7. Find 3 consecutive even numbers such that if they are divided by 2, 4, and 6 respectively, the sum of their quotients will equal the next higher consecutive even number.

Solution: Let x denote the smallest even number. Then we have that

$$\frac{x}{2}+\frac{x+2}{4}+\frac{x+4}{6}=x+6$$
$$6x+3(x+2)+2(x+4)=12(x+6)$$
$$6x+3x+6+2x+8=12x+72$$
$$11x+14=12x+72$$
$$x=-58$$

Therefore, the three consecutive even numbers are -58, -56, and -54.

8. The degree measure of one of two complementary angles is 30 less than twice that of the other. What are the degree measures of the angles?

Solution: Let x denote the degree measure of the first angle and y denote the second. Since the angles are complementary we have that x + y = 90. Further, we have that x = 2y - 30. Plugging the second equation into the first, we have

$$x+y=90$$
$$(2y-30)+y=90$$
$$3y=120$$
$$y=40, x=50$$